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Galerkin domain decomposition procedures for parabolic equations on rectangular domain

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SUMMARY

Galerkin domain decomposition procedures for parabolic equations with three cases of boundary conditions on rectangular domain are discussed. These procedures are non-iterative and non-overlapping ones. They rely on implicit Galerkin method in the sub-domains and integral mean method on the inter-domain boundaries to present explicit flux calculation. Thus, the parallelism can be achieved by the use of these procedures. Two kinds of approximating schemes are presented. Because of the explicit nature of the flux calculation, a less severe time-step constraint is derived to preserve stability. To bound L^2-norm error estimates, new elliptic projections are established and analyzed. Numerical experiments are provided to confirm theoretical results. Copyright © 2009 John Wiley & Sons, Ltd.

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KEY WORDS: domain decomposition procedures; parabolic equations; integral mean method; error estimates; finite element methods; error estimation; parallelization; explicit/implicit; Galerkin

1. INTRODUCTION

When solving parabolic equations, the implicit Galerkin method has an advantage over the explicit Galerkin method. That is, the former is unconditional stable, but the latter needs severe time-step constraint to insure stability. But, a global system of equations must be solved at each time step when using the implicit Galerkin method. On the other hand, most practical problems in engineering can be turned into solving large-scale global partial differential equations, such as reservoir simulation [1, 2], aerodynamics [3], etc. To economize the workload and time, the

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large-scale global equations must be broken into several smaller ones. Parallel algorithms, based upon overlapping or non-overlapping domain decompositions, have been proved to be effective ways. These methods can decompose the large-scale problem into small-scale ones, making the computation more easier (see, e.g. [4–17]).

Dawson and Dupont [9] presented explicit/implicit non-overlapping domain decomposition procedures for parabolic equations. They used implicit Galerkin procedures in the sub-domains and some weight functions on the inter-domain boundary Γ to predict explicit flux calculations. These procedures were conservative both in the sub-domains and across inter-domain boundaries. The explicit nature of the flux calculations induced a time-step limitation to preserve stability, but less severe than that of a fully explicit Galerkin method. They derived *a priori* error bounds in terms of the errors of certain elliptic approximations rather than the powers of some asymptotic parameter. But, in fact, there was a loss of $H^{-1/2}$ factor for space variable, which was noted that it can be avoided in certain special cases using some techniques [18] but was not improved in [9].

Our purpose is to present two kinds of Galerkin nonoverlapping domain decomposition procedures for parabolic equations with three cases of boundary conditions on rectangular domain. These procedures are also explicit/implicit schemes, i.e. implicit Galerkin procedures in the sub-domains and explicit calculations to predict the flux on Γ. But, the ways to predict explicit flux calculations are distinct different from [9]. Firstly, an integral mean method is utilized. The explicit flux calculation is just the integral mean value of the directional derivative of the solution on Γ over a strip domain with width $2H$. This scheme is called as integral mean parallel Galerkin domain decomposition (*IM-PGDD*) scheme. Secondly, in order to improve higher-order accuracy with respect to H, we extrapolate the flux calculation to derive another scheme. We call this case as extrapolation-integral mean parallel Galerkin domain decomposition (*EIM-PGDD*) scheme. Because of the explicit nature of the flux calculation, time-step constraints are derived to preserve the stability that is also less severe than that of a fully explicit Galerkin method. To bound L^2-norm error estimates, new elliptic projections are established and analyzed. With respect to the accuracy order of h, L^2-norm error estimates are optimal for higher-order finite element spaces ($r\geqslant 2$) and almost optimal for linear finite element space ($r=1$). Compared with [9], these L^2-norm error estimates avoid the loss of $H^{-1/2}$ factor.

An outline of this paper is as follows. In Section 2, we introduce the model problem, then present IM-PGDD scheme and EIM-PGDD scheme on rectangular domain, respectively. We discuss three cases of boundary conditions. In Sections 3 and 4, new elliptic projections are defined and analyzed to derive L^2-norm error estimates for IM-PGDD scheme and EIM-PGDD scheme, respectively. In Section 5 we present some numerical experiments, which confirm our theoretical results. Finally, conclusions and perspectives are described in the last section.

Throughout the analysis, the symbols $C, C_1, \tilde{C}_1, \ldots$, etc. will denote generic constants, independent of mesh parameters Δt and h. The symbol C is not necessarily the same at each occurrence. The symbols ε will denote smaller positive constants.

2. GALERKIN DOMAIN DECOMPOSITION PROCEDURES

2.1. Model problem

We will adopt the notations and norms for usual Sobolev spaces. Suppose Ω be a rectangular domain in $\mathscr{R}^2$ with a piecewise uniformly smooth Lipschitz boundary $\partial\Omega$.

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We consider the following parabolic equations:

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \nabla\cdot(A\nabla u) + cu &= f \quad \text{in } \Omega\times(0,T] \\
(A\nabla u)\cdot\mathbf{v} &= 0 \quad \text{on } \partial\Omega\times(0,T] \\
u &= u^0 \quad \text{in } \Omega, \quad t=0
\end{aligned}
\tag{1}
$$

where $\mathbf{v}$ is the unit vector outward normal to $\partial\Omega$, $c=c(t,\mathbf{x})>0$, $u^0=u^0(\mathbf{x})$ and $f=f(t,\mathbf{x})$ are given real-valued functions and assumed as regular as necessary, $A=(a_{ij}(\mathbf{x}))_{2\times 2}$ is a uniformly positive definite matrix function, i.e. there exists positive constants $0<a_0\leqslant a_1$ such that

$$
a_0\sum_{i=1}^{2}\xi_i^2 \leqslant \sum_{i,j=1}^{2} a_{ij}(\mathbf{x})\xi_i\xi_j \leqslant a_1\sum_{i=1}^{2}\xi_i^2 \quad \forall \boldsymbol{\xi}\in\mathscr{R}^2,\ \mathbf{x}\in\Omega \tag{2}
$$

For simplicity, we consider mainly the homogeneous Neumann boundary condition for parabolic equations. The methods discussed below can be easily extended to inhomogeneous Dirichlet or Neumman boundary condition and similar results can be derived. These two cases of boundary conditions are

$$
u = g_1 \quad \text{on } \partial\Omega\times(0,T] \tag{3}
$$

and

$$
(A\nabla u)\cdot\mathbf{v} = g_2 \quad \text{on } \partial\Omega\times(0,T] \tag{4}
$$

respectively. Here, $g_i=g_i(t,\mathbf{x})$ $(i=1,2)$ are smooth given functions. We discuss these two cases briefly later.

2.2. Domain decomposition schemes

We now present Galerkin domain decomposition procedures. Without losing generality, we only discuss the case of unit square domain with two equal sub-domains. But the algorithms and theories can be extended to the case of general rectangular domain with many sub-domains.

Let $\Omega=(0,1)\times(0,1)$ and be divided into two sub-domains $\Omega_i\,(i=1,2)$ by an inter-domain boundary $\Gamma=\{\frac{1}{2}\}\times(0,1)$ (see Figure 1). We denote by $\Gamma_i=\partial\Omega\cap\partial\Omega_i$ the part of the boundary of the sub-domains which coincides with $\partial\Omega$. Let $\mathbf{v}_\Gamma$ denote the unit vector normal to Γ, which points from Ω_1 toward Ω_2.

Let $\mathscr{T}_i^h$ be quasi-uniform partitions of $\Omega_i\,(i=1,2)$ and $\mathscr{T}^h=\mathscr{T}_1^h\cup\mathscr{T}_2^h$, where h denotes the maximal element diameter of $\mathscr{T}^h$. We construct the finite element space $\mathscr{M}^h$ on $\mathscr{T}^h$, which satisfies the following conditions:

(1) For $j=1,2$, let $\mathscr{M}_j^h$ be a finite element subspace of $H^1(\Omega_j)$, and let $\mathscr{M}^h\subset L^2(\Omega)$ such that if $v\in\mathscr{M}^h$, then $v|_{\Omega_j}\in\mathscr{M}_j^h$.
(2) For $j=1,2$, $P_r(\Omega_j)\subset\mathscr{M}_j^h$, where $P_r(\Omega_j)$ is a polynomial space of degree at most r.

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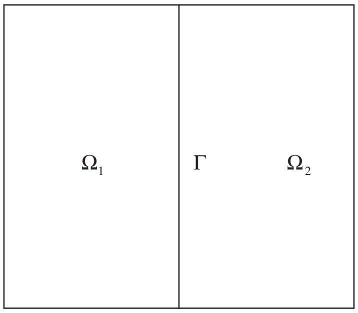

Figure 1. The domain Ω with the inter-domain boundary Γ.

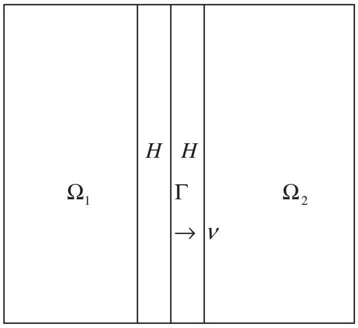

Figure 2. The strip domain $G=[\frac{1}{2}-H,\frac{1}{2}+H]\times[0,1]$ with width $2H$.

(3) For $j=1,2,\ h\in(0,1]$, some integer $k\geqslant 1$ and $u\in H^k(\Omega_j)$, there exists a positive constant C independent of h such that

$$\inf_{v\in\mathscr{M}_j^h}\|u-v\|_{H^s(\Omega_j)}\leqslant Ch^{\sigma}\|u\|_{H^k(\Omega_j)},\quad 0\leqslant s\leqslant 1 \tag{5}$$

where $\sigma=\min(r+1-s,k-s)$.

From the above definition, we note that functions v in $\mathscr{M}^h$ have a well-defined jump $[v]$ on Γ:

$$[v](\mathbf{x})=v(\mathbf{x}^+)-v(\mathbf{x}^-)\quad \forall\mathbf{x} \text{ on } \Gamma$$

where $v(\mathbf{x}^{\pm})\stackrel{\Delta}{=}\lim_{\lambda\to 0^{\pm}}v(\mathbf{x}+\lambda\mathbf{v}_\Gamma)$.

To construct parallel algorithm, for some $H\in(0,\frac{1}{2})$, we introduce an integral mean value of a given function $V\in L^2(\Omega)$ on Γ as (see Figure 2)

$$\overline{V}_H(\mathbf{x})=\frac{1}{2H}\int_{-H}^{H}V(\mathbf{x}+\lambda\mathbf{v}_\Gamma)\,\mathrm{d}\lambda\quad \forall\mathbf{x} \text{ on } \Gamma \tag{6}$$

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Furthermore, the extrapolation of $\overline{V}_H(\mathbf{x})$ on Γ is defined as

$$\widehat{V}_E(\mathbf{x}) = \frac{4\overline{V}_{H/2}(\mathbf{x}) - \overline{V}_H(\mathbf{x})}{3} \quad \forall \mathbf{x} \text{ on } \Gamma \tag{7}$$

Let Δt be time-step size, integer $N = T/\Delta t$, $t^n = n\Delta t$, $\partial_t U^n = (U^n - U^{n-1})/\Delta t, n = 1, \ldots, N$. Introduce a bi-linear form

$$D(\phi, \psi) = (A\nabla\phi, \nabla\psi) + (c\phi, \psi) \quad \forall \phi, \ \psi \in H^1(\Omega)$$

We define the following two parallel Galerkin domain decomposition schemes:

(I) *Integral mean parallel Galerkin domain decomposition scheme (IM-PGDD)*: Given an initial function $U^0 \in \mathscr{M}^h$, seek $\{U^n\}_{n=1}^N \in \mathscr{M}^h$ such that

$$\begin{aligned}(\partial_t U^n, v) &+ D(U^n, v) + (\overline{U}_{\mu,H}^{n-1}, [v])_\Gamma + (\overline{v}_{\mu,H}, [U^{n-1}])_\Gamma + a_1 H^{-1}([U^{n-1}], [v])_\Gamma \\ &= (f^n, v) \quad \forall v \in \mathscr{M}^h \end{aligned} \tag{8}$$

where

$$U_\mu^{n-1} = (A\nabla U^{n-1}) \cdot \mathbf{v}_\Gamma \tag{9}$$

To get the scheme of higher-order accuracy with respect to H, we have another scheme.

(II) *Extrapolation-integral mean parallel Galerkin domain decomposition scheme (EIM-PGDD)*: Given an initial function $U^0 \in \mathscr{M}^h$, seek $\{U^n\}_{n=1}^N \in \mathscr{M}^h$ such that

$$\begin{aligned}(\partial_t U^n, v) &+ D(U^n, v) + (\widehat{U}_{\mu,E}^{n-1}, [v])_\Gamma + (\widehat{v}_{\mu,E}, [U^{n-1}])_\Gamma + 2a_1 H^{-1}([U^{n-1}], [v])_\Gamma \\ &= (f^n, v) \quad \forall v \in \mathscr{M}^h \end{aligned} \tag{10}$$

where

$$U_\mu^{n-1} = (A\nabla U^{n-1}) \cdot \mathbf{v}_\Gamma \tag{11}$$

In these two schemes, the flux on Γ is computed explicitly from U^{n-1} by integral mean method, so that U^n can be computed fully parallel on Ω_1 and Ω_2 once U^{n-1} has been got. These schemes are conservative in the same sense as mentioned in [9].

2.3. Extensions to other boundary conditions

Now, we give a brief discussion for the inhomogeneous Dirichlet boundary condition (3) and Neumman boundary condition (4).

(I) *Inhomogeneous Dirichlet boundary condition* (3) *case*:

$$u = g_1(t, \mathbf{x}) \quad \text{on } \partial\Omega \times (0, T]$$

If $g_1(t, \mathbf{x}) \in L^2(0, T; L^2(\partial\Omega))$, by trace theorem [19], there exists $u_g \in L^2(0, T; H^1(\Omega))$ such that $\gamma_0 u_g|_{\partial\Omega} = g_1$, where $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$ is a continuous trace operator on $\partial\Omega$.

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By setting the transformation $q=u-u_g$, we find q satisfies the following equation:

$$\begin{aligned}\frac{\partial q}{\partial t}-\nabla\cdot(A\nabla q)+cq &= f+\nabla\cdot(A\nabla u_g)-cu_g \quad \text{in } \Omega\times(0,T] \\ q &= 0 \quad \text{on } \partial\Omega\times(0,T] \\ q &= u^0-u_g(0,\mathbf{x}) \quad \text{in } \Omega,\ \ t=0\end{aligned} \tag{12}$$

Then, we only need to present Galerkin domain decomposition procedures for q. The procedures for q are similar to that in Section 2.2, but two modifications should be made.

(1) Firstly, since the boundary condition of Equation (12) is essential, the first condition of element space $\mathscr{M}^h$ in Section 2.2 is modified as follows:

For $j=1,2$, let $\mathscr{M}^h_j$ be a finite element subspace of $H^1(\Omega_j)\cap\{v:v=0 \text{ on } \partial\Omega\}$, and $\mathscr{M}^h\subset L^2(\Omega)$ such that if $v\in\mathscr{M}^h$, then $v|_{\Omega_j}\in\mathscr{M}^h_j$.

(2) Secondly, two terms $((u^n_g)_\nu,[v])_\Gamma$ and $-D(u^n_g,v)$ should be added to the right-hand side of the schemes (8) and (10). For example, the scheme (8) changes to:

Given an initial function $Q^0\in\mathscr{M}^h$, seek $\{Q^n\}^N_{n=1}\in\mathscr{M}^h$ such that

$$\begin{aligned}&(\partial_t Q^n,v)+D(Q^n,v)+(\overline{Q}^{n-1}_{\mu,H},[v])_\Gamma+(\overline{v}_{\mu,H},[Q^{n-1}])_\Gamma+a_1H^{-1}([Q^{n-1}],[v])_\Gamma \\ &\quad=(f^n,v)+((u^n_g)_\mu,[v])_\Gamma-D(u^n_g,v)\quad \forall v\ \in\mathscr{M}^h\end{aligned} \tag{13}$$

where Q^n is the approximating solution of q^n and $(u^n_g)_\mu=(A\nabla u^n_g)\cdot v_\Gamma$.

(II) *Inhomogeneous Neumman boundary condition* (4) *case*:

$$(A\nabla u)\cdot\mathbf{v}=g_2(t,\mathbf{x})\quad \text{on } \partial\Omega\times(0,T]$$

This boundary condition is a natural one. Only one modification should be made. A boundary term $(g_2,v)_{\partial\Omega}$ should be added to the right-hand side of schemes (8) and (10). For example, scheme (8) is as follows:

Given an initial function $Q^0\in\mathscr{M}^h$, seek $\{Q^n\}^N_{n=1}\in\mathscr{M}^h$ such that

$$\begin{aligned}&(\partial_t Q^n,v)+D(Q^n,v)+(\overline{Q}^{n-1}_{\mu,H},[v])_\Gamma+(\overline{v}_{\mu,H},[Q^{n-1}])_\Gamma+a_1H^{-1}([Q^{n-1}],[v])_\Gamma \\ &\quad=(f^n,v)+(g^n_2,v)_{\partial\Omega}\quad \forall v\ \in\mathscr{M}^h\end{aligned} \tag{14}$$

Comparing (13) and (14) with (8), we know that the added terms have no influence on the parallelism of the schemes. The convergence analyses for (13) and (14) are similar to that in the following Sections 3 and 4, respectively. And similar results as in Theorems 3.1 and 4.1 can be derived. We omit them in the rest of the paper.

3. CONVERGENCE ANALYSIS OF IM-PGDD SCHEME

3.1. Basic lemmas

In this section, we will derive convergence theorem for IM-PGDD scheme (8). We need some auxiliary lemmas. Because of the long delay in review for this paper, when this paper was under

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revision, our another paper [20] was accepted for publication, where these auxiliary lemmas were proved similarly. But for the benefit of the readers' reading, we prove the main lemmas here.

First, we have an approximate result on the inter-domain boundary Γ.

Lemma 3.1 (*Ma et al.* [*20*])
For smooth enough function V, there holds estimates

$$\|\overline{V}_H - V\|_{L^2(\Gamma)} \leqslant \sqrt{2H}\|\nabla V\|_{L^2(\Omega)} \tag{15}$$

$$\|\overline{V}_H - V\|_{L^\infty(\Gamma)} \leqslant CH^2\|V\|_{W^{2,\infty}(\Omega)} \tag{16}$$

and

$$V(\mathbf{x}) - \overline{V}_H(\mathbf{x}) = -\tfrac{1}{6}H^2 V_{\nu_\Gamma^2}(\mathbf{x}) - \tfrac{1}{120}H^4 V_{\nu_\Gamma^4}(\mathbf{x}) + o(H^6) \quad \forall \mathbf{x} \text{ on } \Gamma \tag{17}$$

where $V_{\nu_\Gamma^2}$ and $V_{\nu_\Gamma^4}$ are the second and fourth normal derivative of V on Γ, respectively.

To obtain error estimates, we introduce an elliptic projection $V \in \mathscr{M}^h$ of the solution u as follows:

$$(A\nabla(u(\cdot,t) - W(\cdot,t)), \nabla v) + (c(u(\cdot,t) - W(\cdot,t)), v) = 0 \quad \forall v \in \mathscr{M}^h \tag{18}$$

It is clear that the auxiliary problem (18) is equivalent to

$$(A\nabla W, \nabla v) + (cW, v) = \left(f - \frac{\partial u}{\partial t}, v\right) + (-1)^{j+1}(g, v)_\Gamma \quad \forall v \in \mathscr{M}_j^h, \ j = 1, 2 \tag{19}$$

where $g = (A\nabla u)\cdot \mathbf{v}_\Gamma$. These are two standard finite element equations.

Let $\eta = u - W$. From [21–24], we see

Lemma 3.2
For η defined by (18), there holds L^2-norm error estimate

$$\max_{0\leqslant t\leqslant T}\|\eta\|_{L^2(\Omega)} + \|\eta_t\|_{L^2(0,T;L^2(\Omega))} \leqslant Ch^{r+1}\{\|u\|_{L^\infty(0,T;H^{r+1}(\Omega))} + \|u_t\|_{L^2(0,T;H^{r+1}(\Omega))}\} \tag{20}$$

L^∞-norm error estimate

$$\|\eta\|_{L^\infty(\Omega)} + \|\eta_t\|_{L^\infty(\Omega)} \leqslant Ch^2|lnh|\{\|u\|_{W^{2,\infty}(\Omega)} + \|u_t\|_{W^{2,\infty}(\Omega)}\} \quad \text{if } r = 1 \tag{21}$$

$$\|\eta\|_{L^\infty(\Omega)} + \|\eta_t\|_{L^\infty(\Omega)} \leqslant Ch^{r+1}\{\|u\|_{W^{r+1,\infty}(\Omega)} + \|u_t\|_{W^{r+1,\infty}(\Omega)}\} \quad \text{if } r > 1 \tag{22}$$

For functions ψ with restrictions in $H^1(\Omega_1)\cup H^1(\Omega_2)$, we define a norm

$$\||\psi|\|^2 = (A\nabla\psi, \nabla\psi) + a_1 H^{-1}([\psi],[\psi])_\Gamma \tag{23}$$

Lemma 3.3
There exists a positive constant $C_0 = 1 - \frac{\sqrt{2}}{2}$ such that for each $H > 0$,

$$C_0\||\psi|\|^2 \leqslant (A\nabla\psi, \nabla\psi) + a_1 H^{-1}([\psi],[\psi])_\Gamma + 2(\overline{\psi}_{\mu,H}, [\psi])_\Gamma \quad \forall \psi \in \mathscr{M}^h \tag{24}$$

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Proof
Noting that

$$
\begin{aligned}
H\|\overline{\psi}_{\mu,H}\|^2_{L^2(\Gamma)} &= \frac{1}{4H}\int_\Gamma \left[\int_{-H}^{H} |\psi_\mu(\mathbf{x}+\lambda\mathbf{v}_\Gamma)|\,\mathrm{d}\lambda\right]^2 \mathrm{d}\Gamma \\
&= \frac{1}{4H}\int_\Gamma \left[\int_{-H}^{H} |(v_\Gamma^{\mathrm{T}}A^{1/2}v_\Gamma)(v_\Gamma^{\mathrm{T}}A^{1/2}\nabla\psi)(\mathbf{x}+\lambda\mathbf{v}_\Gamma)|\,\mathrm{d}\lambda\right]^2 \mathrm{d}\Gamma \\
&\leqslant \frac{1}{4H}\int_\Gamma \left[\int_{-H}^{H} |(\nabla\psi^\top A\nabla\psi)(\mathbf{x}+\lambda\mathbf{v}_\Gamma)|^{1/2}|\mathbf{v}_\Gamma^\top A\mathbf{v}_\Gamma|^{1/2}\,\mathrm{d}\lambda\right]^2 \mathrm{d}\Gamma \\
&\leqslant \frac{a_1}{2}(A\nabla\psi,\nabla\psi)
\end{aligned}
$$

we have

$$
\begin{aligned}
&(A\nabla\psi,\nabla\psi)+a_1H^{-1}([\psi],[\psi])_\Gamma+2(\overline{\psi}_{\mu,H},[\psi])_\Gamma \\
&\quad\geqslant (A\nabla\psi,\nabla\psi)+a_1H^{-1}\|[\psi]\|^2_{L^2(\Gamma)}-\left[\varepsilon(A\nabla\psi,\nabla\psi)+\frac{a_1}{2\varepsilon}H^{-1}\|[\psi]\|^2_{L^2(\Gamma)}\right] \\
&\quad\geqslant (1-\varepsilon)(A\nabla\psi,\nabla\psi)+\left(1-\frac{1}{2\varepsilon}\right)a_1H^{-1}\|[\psi]\|^2_{L^2(\Gamma)}
\end{aligned}
$$

Here, we used the ε-inequality

$$\alpha\beta\leqslant\varepsilon\alpha^2+\frac{1}{4\varepsilon}\beta^2 \quad \forall\alpha,\beta>0$$

and ε is a smaller arbitrary positive constant. This inequality will be applied frequently in the following analysis. Taking $\varepsilon=\frac{\sqrt{2}}{2}$ in the above estimate leads to (24). The proof of Lemma 3.3 ends. □

3.2. Analysis of a new elliptic projection

The IM-PGDD scheme (8) includes two terms to present explicit flux calculation on the inter-domain boundary Γ by the integral mean method. These two terms are distinct from Dawson–Dupont's schemes and lead to some difficulties in the theoretical analysis. To get optimal error estimates, we need a new elliptic projection $\widetilde{W}\in\mathscr{M}^h$ of the solution u as follows:

$$
\begin{aligned}
&(A\nabla(u-\widetilde{W}),\nabla v)+(\overline{(u-\widetilde{W})}_{\mu,H},[v])_\Gamma+(\overline{v}_{\mu,H},[u-\widetilde{W}])_\Gamma \\
&\quad +a_1H^{-1}([u-\widetilde{W}],[v])_\Gamma=0 \quad \forall v\in\mathscr{M}^h
\end{aligned} \tag{25}
$$

It follows from Lemma 3.3 that the project problem (25) has a unique solution for H.

Let

$$\xi^n=u^n-\widetilde{W}^n, \quad \theta^n=U^n-\widetilde{W}^n \tag{26}$$

The following lemma gives the bounds of ξ.

Lemma 3.4 (*Ma et al.* [*20*])
There holds the *a priori* estimates:

$$\|\xi\|_{L^2(\Omega)} \leqslant C\{h^{r+1} + H^{1/2}\|\eta\|_{L^\infty(\Omega)}\} \tag{27}$$

and

$$\|\xi_t\|_{L^2(\Omega)} \leqslant C\{h^{r+1} + H^{1/2}\|\eta_t\|_{L^\infty(\Omega)}\} \tag{28}$$

3.3. Error estimate

Now, we turn to derive an $L^2(\Omega)$-norm error estimate for θ^n. It follows from trace theorem [25] that

$$\|\psi\|^2_{L^2(\Gamma)} \leqslant C_2\|\psi\|\|\psi\|_1 \quad \forall \psi \in H^1(\Omega) \tag{29}$$

which will be used in the following proof.

Lemma 3.5
For θ^n defined in (26), there holds the following error estimate:

$$\max_{0\leqslant n\leqslant N} \|\theta^n\|^2_{L^2(\Omega)} \leqslant C\left\{H\left[\max_{1\leqslant n\leqslant N}(\|\eta^n\|^2_{L^\infty(\Omega)} + \|\eta^n\|^2) + \|\eta_t\|^2_{L^2(0,T;L^2(\Omega))}\right] + (\Delta t)^2 + H^5 + h^{2(r+1)}\right\} \tag{30}$$

provided

$$\Delta t \leqslant C_1 H^2, \quad C_1 = \frac{a_0(1-\delta)^2 C_0^3}{16 a_1^2 C_2^2} \quad \forall 0<\delta\ll 1 \tag{31}$$

Proof
From (1), we get the weak formulation:

$$(\partial_t u^n, v) + D(u^n, v) + (u^n_\mu, [v])_\Gamma = (f^n + \rho^n, v) \quad \forall v \in \mathscr{M}^h \tag{32}$$

where $u^n(x) = u(x, t^n)$ and $u^n_\mu = (A\nabla u^n)\cdot \mathbf{v}_\Gamma$. The time truncation term ρ^n satisfies

$$\sum_{n=1}^{M} \|\rho^n\|^2 \Delta t^n \leqslant (\Delta t)^2 \int_0^{\mathrm{T}} \|u_{tt}(\cdot, t)\|^2 \,\mathrm{d}t \leqslant C(\Delta t)^2$$

Combining (25) and (32), we have

$$\begin{aligned}
&(\partial_t \widetilde{W}^n, v) + (A\nabla\widetilde{W}^n, \nabla v) + (\overline{\widetilde{W}}^{n-1}_{\mu,H}, [v])_\Gamma + (\overline{v}_{\mu,H}, [\widetilde{W}^{n-1}])_\Gamma + a_1 H^{-1}([\widetilde{W}^{n-1}], [v])_\Gamma \\
&\quad = (\overline{\widetilde{W}}^{n-1}_{\mu} - \overline{\widetilde{W}}^{n}_{\mu}, [v])_\Gamma + (\overline{v}_{\mu,H}, [\widetilde{W}^{n-1} - \widetilde{W}^n])_\Gamma + a_1 H^{-1}([\widetilde{W}^{n-1} - \widetilde{W}^n], [v])_\Gamma \\
&\qquad + (f^n + \rho^n, v) - (\partial_t \xi^n + c u^n, v) + (\overline{u}^n_{\mu,H} - u^n_\mu, [v])_\Gamma \quad \forall v \in \mathscr{M}^h
\end{aligned} \tag{33}$$

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Subtracting (33) from (8) and taking $v=\theta^n$, we obtain

$$
\begin{aligned}
&(\partial_t\theta^n,\theta^n)+(A\nabla\theta^n,\nabla\theta^n)+a_1H^{-1}([\theta^n],[\theta^n])_\Gamma+2(\overline{\theta}^n_{\mu,H},[\theta^n])_\Gamma\\
&\quad=(\overline{\theta}^n_{\mu,H}-\overline{\theta}^{n-1}_{\mu,H},[\theta^n])_\Gamma+(\overline{\theta}^n_{\mu,H},[\theta^n-\theta^{n-1}])_\Gamma+a_1H^{-1}([\theta^n-\theta^{n-1}],[\theta^n])_\Gamma\\
&\qquad+(A\nabla(\xi^n-\xi^{n-1}),\nabla\theta^n)+(\partial_t\xi^n+c\xi^n-c\theta^n-\rho^n,\theta^n)\\
&\qquad-(\overline{u}^{n-1}_{\mu,H}-\overline{u}^n_{\mu,H},[\theta^n])_\Gamma+(u^n_\mu-\overline{u}^n_{\mu,H},[\theta^n])_\Gamma
\end{aligned}\tag{34}
$$

Since

$$
(\partial_t\theta^n,\theta^n)=\frac{1}{2\Delta t}[\|\theta^n\|^2-\|\theta^{n-1}\|^2]+\frac{\Delta t}{2}\|\partial_t\theta^n\|^2
$$

and

$$
(\overline{\theta}^n_{\mu,H},[\theta^n-\theta^{n-1}])_\Gamma=(\overline{\theta}^n_{\mu,H},[\theta^n])_\Gamma-(\overline{\theta}^{n-1}_{\mu,H},[\theta^{n-1}])_\Gamma-(\overline{\theta}^n_{\mu,H}-\overline{\theta}^{n-1}_{\mu,H},[\theta^{n-1}])_\Gamma
$$

summing (34) over n yields

$$
\begin{aligned}
&\frac{1}{2}\|\theta^n\|^2+\frac{(\Delta t)^2}{2}\|\partial_t\theta^n\|^2+\Delta t[(A\nabla\theta^n,\nabla\theta^n)+a_1H^{-1}([\theta^n],[\theta^n])_\Gamma+(\overline{\theta}^n_{\mu,H},[\theta^n])_\Gamma]\\
&\qquad+\Delta t\sum_{k=1}^{n-1}\left[\frac{\Delta t}{2}\|\partial_t\theta^k\|^2+(A\nabla\theta^k,\nabla\theta^k)+a_1H^{-1}([\theta^k],[\theta^k])_\Gamma+2(\overline{\theta}^k_{\mu,H},[\theta^k])_\Gamma\right]\\
&\quad=\frac{1}{2}\|\theta^0\|^2-\Delta t(\overline{\theta}^0_{\mu,H},[\theta^0])_\Gamma+\Delta t\sum_{k=1}^{n}[(\overline{\theta}^k_{\mu,H}-\overline{\theta}^{k-1}_{\mu,H},[\theta^k-\theta^{k-1}])_\Gamma\\
&\qquad+a_1H^{-1}([\theta^k-\theta^{k-1}],[\theta^k])_\Gamma+(A\nabla(\xi^k-\xi^{k-1}),\nabla\theta^k)+(\partial_t\xi^k+c\xi^k-c\theta^k-\rho^k,\theta^k)\\
&\qquad-(\overline{u}^{k-1}_{\mu,H}-\overline{u}^k_{\mu,H},[\theta^k])_\Gamma+(u^k_\mu-\overline{u}^k_{\mu,H},[\theta^k])_\Gamma]
\end{aligned}\tag{35}
$$

Furthermore, noting that

$$
2(\overline{\theta}^k_{\mu,H},[\theta^k])_\Gamma\leqslant\varepsilon\|\theta^k\|^2_{1,A}+\frac{a_1}{2\varepsilon}H^{-1}\|[\theta^k]\|^2_{L^2(\Gamma)}
$$

where $\|v\|^2_{1,A}=(A\nabla v,\nabla v)$, and taking $0<\varepsilon=\frac{\sqrt{2}}{2}<1$, we have

$$
\begin{aligned}
&\frac{1}{2}\|\theta^n\|^2+\frac{\Delta t}{2}(1-C_0)((A\nabla\theta^n,\nabla\theta^n)+a_1H^{-1}\|[\theta^n]\|^2_{L^2(\Gamma)})\\
&\qquad+\Delta t\sum_{k=1}^{n}\left[\frac{\Delta t}{2}\|\partial_t\theta^k\|^2+C_0((A\nabla\theta^k,\nabla\theta^k)+a_1H^{-1}\|[\theta^k]\|^2_{L^2(\Gamma)})\right]\\
&\quad\leqslant\frac{1}{2}\|\theta^0\|^2+\Delta t|(\overline{\theta}^0_{\mu,H},[\theta^0])_\Gamma|+\Delta t\sum_{k=1}^{n}[(\overline{\theta}^k_{\mu,H}-\overline{\theta}^{k-1}_{\mu,H},[\theta^k-\theta^{k-1}])_\Gamma+a_1H^{-1}([\theta^k-\theta^{k-1}],[\theta^k])_\Gamma\\
&\qquad+(A\nabla(\xi^k-\xi^{k-1}),\nabla\theta^k)+(\partial_t\xi^k+c\xi^k-c\theta^k-\rho^k,\theta^k)-(\overline{u}^{k-1}_{\mu,H}-\overline{u}^k_{\mu,H},[\theta^k])_\Gamma\\
&\qquad+(u^k_\mu-\overline{u}^k_{\mu,H},[\theta^k])_\Gamma]
\end{aligned}\tag{36}
$$

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We estimate the terms on the right-hand side of (36) one by one. From (29), we see that

$$
\begin{aligned}
\Delta t \sum_{k=1}^{n} (\overline{\theta}_{\mu,H}^{k} - \overline{\theta}_{\mu,H}^{k-1}, [\theta^k - \theta^{k-1}])_\Gamma &\leqslant \Delta t \sum_{k=1}^{n} \|[\theta^k - \theta^{k-1}]\|_{L^2(\Gamma)} \|\overline{\theta}_{\mu,H}^{k} - \overline{\theta}_{\mu,H}^{k-1}\|_{L^2(\Gamma)} \\
&\leqslant 2C_2^{1/2} \Delta t \sum_{k=1}^{n} \|\theta^k - \theta^{k-1}\|_1^{1/2} \|\theta^k - \theta^{k-1}\|^{1/2} \|\overline{\theta}_{\mu,H}^{k} - \overline{\theta}_{\mu,H}^{k-1}\|_{L^2(\Gamma)} \\
&\leqslant \frac{1}{2} (a_0 C_0 \Delta t)^{1/2} \sum_{k=1}^{n} (\|\theta^k\|_1 + \|\theta^{k-1}\|_1) \|\theta^k - \theta^{k-1}\| + \frac{2C_2 (\Delta t)^{3/2}}{\sqrt{a_0 C_0}} \sum_{k=1}^{n} \frac{a_1}{2H} \|\theta^k - \theta^{k-1}\|_{1,A}^2 \\
&\leqslant \frac{(\Delta t)^2}{4} \sum_{k=1}^{n} \|\partial_t \theta^k\|^2 + \frac{a_0 C_0 \Delta t}{2} \sum_{k=1}^{n} \|\theta^k\|_1^2 + \frac{4a_1 C_2 (\Delta t)^{3/2}}{\sqrt{a_0 C_0} H} \sum_{k=1}^{n} \|\theta^k\|_{1,A}^2 \\
&\leqslant \frac{(\Delta t)^2}{4} \sum_{k=1}^{n} \|\partial_t \theta^k\|^2 + \frac{C_0 \Delta t}{2} \sum_{k=1}^{n} [\|\theta^k\|_{1,A}^2 + a_0 \|\theta^k\|^2] + (1-\delta) C_0 \Delta t \sum_{k=1}^{n} \|\theta^k\|_{1,A}^2
\end{aligned} \tag{37}
$$

provided that

$$
\Delta t \leqslant \frac{a_0 (1-\delta)^2 C_0^3 H^2}{16 a_1^2 C_2^2} \quad \forall 0 < \delta \ll 1 \tag{38}
$$

Similarly, we have

$$
\begin{aligned}
\frac{a_1 \Delta t}{H} \sum_{k=1}^{n} ([\theta^k - \theta^{k-1}], [\theta^k])_\Gamma
&\leqslant \frac{a_1 \Delta t}{H} \sum_{k=1}^{n} \|[\theta^k - \theta^{k-1}]\|_{L^2(\Gamma)} \|[\theta^k]\|_{L^2(\Gamma)} \\
&\leqslant \frac{2a_1 C_2^{1/2} \Delta t}{H} \sum_{k=1}^{n} \|\theta^k - \theta^{k-1}\|_1^{1/2} \|\theta^k - \theta^{k-1}\|^{1/2} \|[\theta^k]\|_{L^2(\Gamma)}
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
&\leqslant \frac{1}{2} (a_0 C_0 \Delta t)^{1/2} \sum_{k=1}^{n} (\|\theta^k\|_1 + \|\theta^{k-1}\|_1) \|\theta^k - \theta^{k-1}\| + \frac{2a_1^2 C_2 (\Delta t)^{3/2}}{\sqrt{a_0 C_0} H^2} \sum_{k=1}^{n} \|[\theta^k]\|_{L^2(\Gamma)}^2 \\
&\leqslant \frac{(\Delta t)^2}{4} \sum_{k=1}^{n} \|\partial_t \theta^k\|^2 + \frac{a_0 C_0 \Delta t}{2} \sum_{k=1}^{n} \|\theta^k\|_1^2 + \frac{2a_1^2 C_2 (\Delta t)^{3/2}}{\sqrt{a_0 C_0} H^2} \sum_{k=1}^{n} \|[\theta^k]\|_{L^2(\Gamma)}^2 \\
&\leqslant \frac{(\Delta t)^2}{4} \sum_{k=1}^{n} \|\partial_t \theta^k\|^2 + \frac{C_0 \Delta t}{2} \sum_{k=1}^{n} [\|\theta^k\|_{1,A}^2 + a_0 \|\theta^k\|^2] \\
&\quad + (1-\delta) C_0 \Delta t a_1 H^{-1} \sum_{k=1}^{n} \|[\theta^k]\|_{L^2(\Gamma)}^2
\end{aligned} \tag{40}
$$

provided that

$$
\Delta t \leqslant \frac{a_0 (1-\delta)^2 C_0^3 H^2}{4 a_1^2 C_2^2} \quad \forall 0 < \delta \ll 1 \tag{41}
$$

It is easy to get

$$|(A\nabla(\xi^k-\xi^{k-1}),\nabla\theta^k)|\leqslant\frac{\Delta t}{2}[\|\theta^k\|_{1,A}^2+\|\partial_t\xi^k\|_{1,A}^2] \tag{42}$$

For the last three terms, we can get

$$|(\partial_t\xi^k+c\xi^k-c\theta^k-\rho^k,\theta^k)|\leqslant K\{\|\rho^k\|^2+\|\xi^k\|^2+\|\partial_t\xi^k\|^2+\|\theta^k\|^2\} \tag{43}$$

$$\begin{aligned}|(\overline{u}_{\mu,H}^k-\overline{u}_{\mu,H}^{k-1},[\theta^k])_\Gamma| &\leqslant \frac{H}{2\delta C_0 a_1}\|\overline{u}_{\mu,H}^k-\overline{u}_{\mu,H}^{k-1}\|_{L^2(\Gamma)}^2+0.5\delta C_0 a_1 H^{-1}\|[\theta^k]\|_{L^2(\Gamma)}^2\\ &\leqslant C\|u^k-u^{k-1}\|_{1,A}^2+0.5\delta C_0 a_1 H^{-1}\|[\theta^k]\|_{L^2(\Gamma)}^2\end{aligned} \tag{44}$$

and

$$\begin{aligned}|(u_\mu^k-\overline{u}_{\mu,H}^k,[\theta^k])_\Gamma| &\leqslant \frac{H}{2\delta C_0 a_1}\|u_\mu^k-\overline{u}_{\mu,H}^k\|_{L^\infty(\Gamma)}^2+0.5\delta C_0 a_1 H^{-1}\|[\theta^k]\|_{L^2(\Gamma)}^2\\ &\leqslant CH^5\|u^k\|_{W^{2,\infty}(\Omega)}^2+0.5\delta C_0 a_1 H^{-1}\|[\theta^k]\|_{L^2(\Gamma)}^2\end{aligned} \tag{45}$$

by Lemma 3.1.

By (38) and (41), we take

$$\Delta t=\frac{a_0(1-\delta)^2C_0^3H^2}{16a_1^2C_2^2}\overset{Def.}{=}C_1H^2\quad \forall 0<\delta\ll 1 \tag{46}$$

Collecting together from (36) to (45), we obtain

$$\begin{aligned}&\frac{1}{2}\|\theta^n\|^2+\frac{\Delta t}{2}(1-C_0)(\|\theta^n\|_{1,A}^2+a_1H^{-1}\|[\theta^n]\|_{L^2(\Gamma)}^2)\\ &\quad\leqslant\frac{1}{2}\|\theta^0\|^2+\Delta t|(\overline{\theta}_{\mu,H}^0,[\theta^0])_\Gamma|+C\Delta t\sum_{k=1}^n[\Delta t(\|\theta^k\|_{1,A}^2+\|\partial_t\xi^k\|_{1,A}^2)+\|\theta^k\|^2\\ &\qquad+\|\partial_t\xi^k\|^2+\|\xi^k\|+\|\rho^k\|^2+(\Delta t)^2\|\nabla u_t^k\|^2+H^5\|u^k\|_{W^{2,\infty}(\Omega)}^2]\end{aligned} \tag{47}$$

Taking $\theta^0=0$, using the discrete Gronwall lemma and Lemma 3.4 to (47), we finally drive (30). □

Now, we can derive an error estimate as the following.

Theorem 3.1

Let u and $\{U^n\}_{n=1}^N$ be the solutions of parabolic equations (1) and IM-PGDD scheme (8), respectively. Suppose that u is sufficiently smooth and $U^0\in\mathscr{M}^h$ is taken to be $\widetilde{W}^0$, which is defined by (25). For linear finite element spaces, let $H=O((1+|\ln h|)^{-2})$. Then there exists a positive constant C independent of mesh sizes h, H and Δt, such that

$$\max_{1\leqslant n\leqslant N}\|u^n-U^n\|_{L^2(\Omega)}\leqslant C\{\Delta t+h^{r+1}+H^{5/2}\} \tag{48}$$

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provided

$$\Delta t \leqslant C_1 H^2 \tag{49}$$

where constant C_1 is given by (31).

Proof
Applying Lemmas 3.4 and 3.5, we have

$$\begin{aligned}\max_{1\leqslant n\leqslant N} \|u^n - U^n\| &\leqslant \max_{1\leqslant n\leqslant N} \{\|\xi^n\| + \|\theta^n\|\} \\ &\leqslant C\{h^{r+1} + H^{1/2} \max_{1\leqslant n\leqslant N} \|\eta^n\|_{L^\infty(\Omega)} + H^{1/2}\|\eta_t\|_{L^2(0,T;L^2(\Omega))} + \Delta t + H^{5/2}\}\end{aligned} \tag{50}$$

By (50), using (21) and (22), respectively, we have

(1) For linear finite element space ($r=1$) case

$$\begin{aligned}\max_{1\leqslant n\leqslant N} \|u^n - U^n\| &\leqslant C\{h^2 + H^{1/2}h^2|lnh| + H^{1/2}h^2 + \Delta t + H^{5/2}\} \\ &\leqslant C\{h^2 + H^{1/2}h^2(1+|lnh|) + \Delta t + H^{5/2}\} \\ &\leqslant C\{h^2 + \Delta t + H^{5/2}\}\end{aligned} \tag{51}$$

provided that $H = O((1+|lnh|)^{-2})$.

(2) For the other finite element space ($r>1$) case

$$\begin{aligned}\max_{1\leqslant n\leqslant N} \|u^n - U^n\| &\leqslant C\{h^{r+1} + H^{1/2}h^{r+1} + \Delta t + H^{5/2}\} \\ &\leqslant C\{h^{r+1} + \Delta t + H^{5/2}\}\end{aligned} \tag{52}$$

Then we can derive (48). □

Remark 1
For linear finite element spaces, i.e. $r=1$, Theorem 3.1 shows a weaker restraint of mesh ratio between Δt and h than that of explicit Galerkin schemes. In explicit Galerkin schemes, mesh ratio $\Delta t/h^2 \leqslant C^*$ is necessary, where some constant $C^* > 0$. But in our IM-PGDD scheme, if we take $H^{5/2} = O(h^2)$ to balance error accuracy with respect to h and H, only mesh ratio $\Delta t/h^{8/5} \leqslant C_1$ is required for some constant $C_1 > 0$. It is easy to see that mesh ratio of IM-PGDD scheme is $h^{-2/5}$ multiple of that of explicit Galerkin schemes.

From Theorem 3.1, we know that the L^2-norm error estimate is optimal for higher-order finite element spaces ($r \geqslant 2$) and almost optimal for linear finite element space ($r=1$) with respect to the accuracy order of h.

4. CONVERGENCE ANALYSIS OF EIM-PGDD SCHEME

Since EIM-PGDD scheme (10) extrapolates the flux calculation of IM-PGDD scheme (8) on the inter-domain boundary Γ, the convergence analysis of scheme (10) is similar to that of scheme (8). We will describe the processes of analysis and present Theorem 4.1 simply. Theorem 4.1 is also based on some basic lemmas, whose proofs are similar to that of Lemmas in Section 3 accordingly.

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Lemma 4.1 (*Ma et al.* [*20*])
For smooth enough function V, there holds estimates

$$\|\widehat{V}_E - V\|_{L^2(\Gamma)} \leqslant \frac{2\sqrt{2}+1}{3}\sqrt{2H}\|\nabla V\|_{L^2(\Omega)} \tag{53}$$

$$\|\widehat{V}_E - V\|_{L^\infty(\Gamma)} \leqslant C H^4 \|V\|_{W^{4,\infty}(\Omega)} \tag{54}$$

and

$$V(\mathbf{x}) - \widehat{V}_E(\mathbf{x}) = \tfrac{1}{480} H^4 V_{\nu_\Gamma^4}(\mathbf{x}) + o(H^6) \quad \forall \mathbf{x} \text{ on } \Gamma \tag{55}$$

where $V_{\nu_\Gamma^4}$ is the fourth normal derivative of V on Γ.

For functions ψ with restrictions in $H^1(\Omega_1) \cup H^1(\Omega_2)$, we use the norm

$$\||\psi|\|^2 = (A\nabla\psi, \nabla\psi) + 2a_1 H^{-1}([\psi],[\psi])_\Gamma \tag{56}$$

Lemma 4.2
There exists a positive constant $\tilde{C}_0 = 1 - 2\sqrt{2}/3$ such that for each $H > 0$,

$$\tilde{C}_0 \||\psi|\|^2 \leqslant (A\nabla\psi, \nabla\psi) + 2a_1 H^{-1}([\psi],[\psi])_\Gamma + 2(\widehat{\psi}_{\mu,E}, [\psi])_\Gamma \quad \forall \psi \in \mathscr{M}^h \tag{57}$$

Proof
Noting that

$$\begin{aligned}
\widehat{\psi}_{\mu,E} &= \frac{4}{3H}\int_{-H/2}^{H/2} \psi_\mu(\mathbf{x}+\lambda\mathbf{v}_\Gamma)\,\mathrm{d}\lambda - \frac{1}{6H}\int_{-H}^{H} \psi_\mu(\mathbf{x}+\lambda\mathbf{v}_\Gamma)\,\mathrm{d}\lambda \\
&= \frac{1}{H}\int_{-H/2}^{H/2} \psi_\mu(\mathbf{x}+\lambda\mathbf{v}_\Gamma)\,\mathrm{d}\lambda - \frac{1}{6H}\int_{|\lambda|\geqslant H/2} \psi_\mu(\mathbf{x}+\lambda\mathbf{v}_\Gamma)\,\mathrm{d}\lambda \\
&\quad + \frac{1}{6H}\int_{|\lambda|\leqslant H/2} \psi_\mu(\mathbf{x}+\lambda\mathbf{v}_\Gamma)\,\mathrm{d}\lambda \\
&\leqslant \frac{4}{3\sqrt{H}}\left[\int_{-H}^{H} |\psi_\mu(\mathbf{x}+\lambda\mathbf{v}_\Gamma)|^2\,\mathrm{d}\lambda\right]^{1/2}
\end{aligned}$$

such that

$$H\|\widehat{\psi}_{\mu,E}\|^2_{L^2(\Gamma)} \leqslant \frac{16}{9}\int_\Gamma\int_{-H}^{H} |\psi_\mu(\mathbf{x}+\lambda\mathbf{v}_\Gamma)|^2\,\mathrm{d}\lambda\,\mathrm{d}\Gamma \leqslant \frac{16a_1}{9}\|\psi\|^2_{1,A}$$

We have

$$\begin{aligned}
&(A\nabla\psi, \nabla\psi) + 2a_1 H^{-1}([\psi],[\psi])_\Gamma + 2(\widehat{\psi}_{\mu,E}, [\psi])_\Gamma \\
&\quad \geqslant (1-\varepsilon)(A\nabla\psi, \nabla\psi) + 2\left(1 - \frac{8}{9\varepsilon}\right) a_1 H^{-1} \|[\psi]\|^2_{L^2(\Gamma)}
\end{aligned}$$

Taking $\varepsilon = 2\sqrt{2}/3$ in the above estimate leads to (57). The proof of Lemma 4.2 ends. □

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To get error estimates, we introduce an elliptic projection $\widetilde{W}\in\mathscr{M}^h$ of the solution u as follows:

$$(A\nabla(u-\widetilde{W}),\nabla v)+(\widehat{(u-\widetilde{W})}_{\mu,E},[v])_\Gamma+(\widehat{v}_{\mu,E},[u-\widetilde{W}])_\Gamma$$

$$+2a_1H^{-1}([u-\widetilde{W}],[v])_\Gamma=0\quad\forall v\in\mathscr{M}^h \tag{58}$$

It follows from Lemma 4.2 that the project problem (58) has unique solution for H.

Let

$$\xi^n=u^n-\widetilde{W}^n,\quad \theta^n=U^n-\widetilde{W}^n \tag{59}$$

The following lemma gives the bounds of ξ.

Lemma 4.3
There holds the *a priori* estimates:

$$\|\xi\|_{L^2(\Omega)}\leqslant C\{h^{r+1}+H^{1/2}\|\eta\|_{L^\infty(\Omega)}\} \tag{60}$$

and

$$\|\xi_t\|_{L^2(\Omega)}\leqslant C\{h^{r+1}+H^{1/2}\|\eta_t\|_{L^\infty(\Omega)}\} \tag{61}$$

Now, we can derive an $L^2(\Omega)$-norm error estimate for θ^n.

Lemma 4.4
For θ^n defined in (59), there holds the following error estimate

$$\max_{0\leqslant n\leqslant N}\|\theta^n\|^2_{L^2(\Omega)}\leqslant C\left\{H\left[\max_{1\leqslant n\leqslant N}(\|\eta^n\|^2_{L^\infty(\Omega)}+\|\eta^n\|^2)+\|\eta_t\|^2_{L^2(0,T;L^2(\Omega))}\right]\right.$$

$$\left.+(\Delta t)^2+H^9+h^{2(r+1)}\right\} \tag{62}$$

provided

$$\Delta t\leqslant\tilde{C}_1H^2,\quad \tilde{C}_1=\frac{a_0(1-\delta)^2\tilde{C}_0^3}{16a_1^2C_2^2}\quad\forall 0<\delta\ll 1 \tag{63}$$

Here, the main equation for θ is

$$(\partial_t\theta^n,v)+(A\nabla\theta^n,\nabla v)+(\widehat{\theta}^n_{\mu,E},[v])_\Gamma+(\widehat{v}_{\mu,E},[\theta^n])_\Gamma+2a_1H^{-1}([\theta^n],[v])_\Gamma$$

$$=(\widehat{\theta}^n_{\mu,E}-\widehat{\theta}^{n-1}_{\mu,E},[v])_\Gamma+(\widehat{v}_{\mu,E},[\theta^n-\theta^{n-1}])_\Gamma+2a_1H^{-1}([\theta^n-\theta^{n-1}],[v])_\Gamma$$

$$+(A\nabla(\xi^n-\xi^{n-1}),\nabla v)+(\partial_t\xi^n+c\xi^n-c\theta^n-\rho^n,v)$$

$$-(\widehat{u}^{n-1}_{\mu,E}-\widehat{u}^n_{\mu,E},[v])_\Gamma+(u^n_\mu-\widehat{u}^n_{\mu,E},[v])_\Gamma \tag{64}$$

Similar to the proofs of Theorem 3.1, applying Lemmas 4.3 and 4.4, we can derive the following theorem.

Theorem 4.1
Let u and $\{U^n\}_{n=1}^N$ be the solutions of parabolic equations (1) and EIM-PGDD scheme (10), respectively. Suppose that u is sufficiently smooth and $U^0 \in \mathscr{M}^h$ is taken to be $\widetilde{W}^0$, which is defined by (58). For linear finite element spaces, let $H = O((1+|\ln h|)^{-2})$. Then there exists a positive constant C independent of mesh sizes h, H and Δt, such that

$$\max_{1\leqslant n\leqslant N} \|u^n - U^n\|_{L^2(\Omega)} \leqslant C\{\Delta t + h^{r+1} + H^{9/2}\} \tag{65}$$

provided

$$\Delta t \leqslant \tilde{C}_1 H^2 \tag{66}$$

where constant $\tilde{C}_1$ is given by (63).

Remark 2
Analogically as Remark 1, for linear finite element spaces, i.e. $r=1$, Theorem 4.1 also shows a weaker restraint of mesh ratio between Δt and h than that of explicit Galerkin schemes. In the EIM-PGDD scheme, if we take $H^{9/2} = O(h^2)$ to balance error accuracy, only the condition $\Delta t / h^{8/9} \leqslant \tilde{C}_1$ is needed for some constant $\tilde{C}_1 > 0$. The mesh ratio of EIM-PGDD scheme is $h^{-10/9}$ multiple of that of explicit Galerkin schemes.

From Theorem 4.1, we know that the L^2-norm error estimate is optimal for higher-order finite element spaces ($r \geqslant 2$) and almost optimal for linear finite element space ($r=1$) with respect to the accuracy order of h.

Comparing Theorem 3.1 with Theorem 4.1, we can see that the EIM-PGDD scheme has an accuracy of higher order for H better than that of the IM-PGDD scheme, and has the accuracy of the same order as that of the implicit method on Δt and h. This shows that the EIM-PGDD scheme can use larger H than that of IM-PGDD scheme so that the time-step constraint of the EIM-PGDD scheme is weaker than the IM-PGDD scheme.

5. NUMERICAL EXPERIMENTS

In this section, we present some numerical experiments for the schemes described above.

Example 1 (*IM-PGDD scheme case*)**.**
We consider the following case of two sub-domains: let $\Omega = (0,1)\times(0,1)$, and take $\Omega_1 = (0, \frac{1}{2}) \times (0,1)$, $\Omega_2 = (\frac{1}{2}, 1)\times(0,1)$, and the inter-domain boundary $\Gamma = \{\frac{1}{2}\}\times(0,1)$. The problem is just as same as the second model in [9]:

$$\begin{aligned} u_t - \Delta u &= f(x,y,t) \quad \text{in } \Omega\times(0,T] \\ u^0(x,y) &= 0 \quad \text{in } \Omega,\ t=0 \end{aligned} \tag{67}$$

where $f(x,y,t)$ is chosen so that $u(x,y,t) = 100tx^3(1-x)^2\cos(2\pi y)$. See Figures 3 and 4.

In these runs, the solution u is approximated in the space of piecewise continuous bilinear function. We consider two scenarios: (1) fully implicit Galerkin method on uniform mesh; i.e. no

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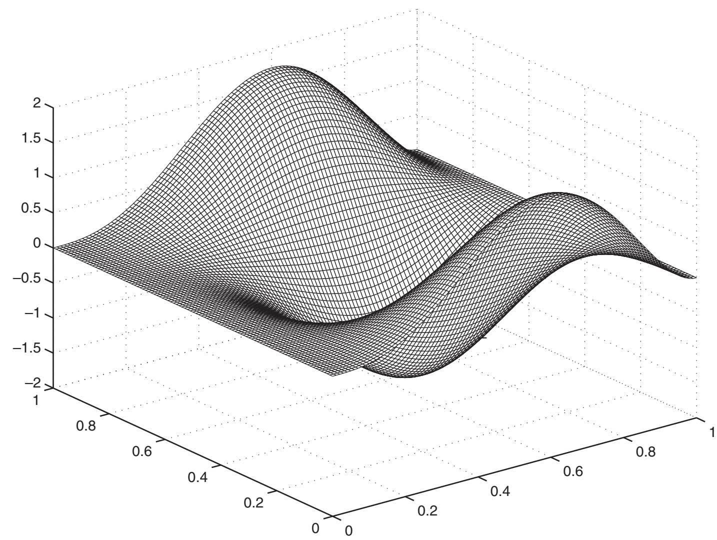

Figure 3. The solution $u(x, y, t) = 100tx^3(1-x)^2\cos(2\pi y)$ at $t = 0.5$.

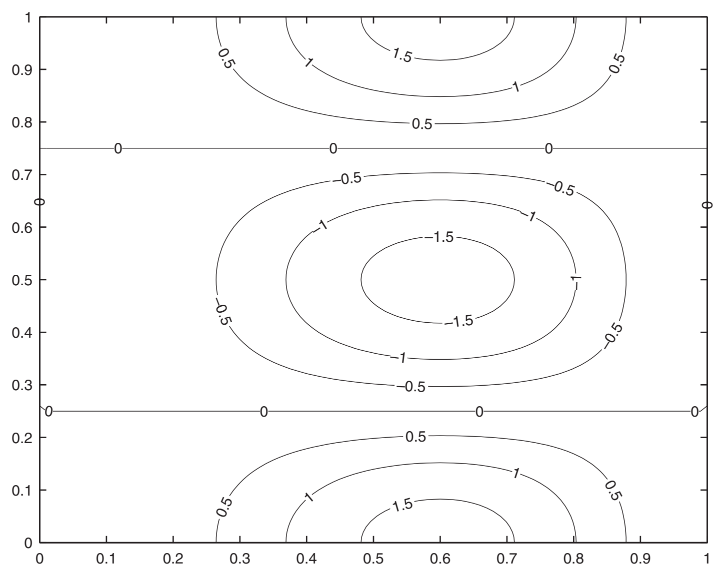

Figure 4. The contour of the solution $u(x, y, t) = 100tx^3(1-x)^2\cos(2\pi y)$ at $t = 0.5$.

domain decomposition; (2) Galerkin domain decomposition with two sub-domains, with interface $\Gamma = \{\frac{1}{2}\} \times (0, 1)$ and global uniform mesh.

We approximate (67) by the IM-PGDD scheme using 4-node quadrilateral mesh on $20\times 20, 40\times 40, 80\times 80, 160\times 160$ and 320×320 grids, respectively. In these runs, we take $H^{5/2} = h^2$ to balance error accuracy with respect to h and H. From Remark 1, we know that

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Table I. L^2-norm error for each case at $t=0.5$.

			Implicit		IM-PGDD	
Grids	h	H/h	$\|e_h\|_{L^2}$	Rate	$\|e_h\|_{L^2}$	Rate
20×20	$0.5000e-01$	$0.1821e+01$	$0.8065e-02$		$0.7942e-02$	
40×40	$0.2500e-01$	$0.2091e+01$	$0.2020e-02$	1.997	$0.1985e-02$	2.000
80×80	$0.1250e-01$	$0.2402e+01$	$0.5050e-03$	2.000	$0.4962e-03$	2.000
160×160	$0.6250e-02$	$0.2759e+01$	$0.1264e-03$	1.998	$0.1241e-03$	2.000
320×320	$0.3125e-02$	$0.3168e+01$	$0.3154e-04$	2.003	$0.3103e-04$	2.000

Table II. L^∞-norm error of grids on Γ at $t=0.5$.

			Implicit		IM-PGDD			
Grids	h	H/h		Rate	L-side	Rate	R-side	Rate
20×20	$0.5000e-01$	$0.1821e+01$	$0.6323e-02$		$0.5324e-02$		$0.5346e-02$	
40×40	$0.2500e-01$	$0.2091e+01$	$0.1574e-02$	2.006	$0.1363e-02$	1.966	$0.1377e-02$	1.957
80×80	$0.1250e-01$	$0.2402e+01$	$0.3721e-03$	2.081	$0.3348e-03$	2.025	$0.3364e-03$	2.033
160×160	$0.6250e-02$	$0.2759e+01$	$0.9018e-04$	2.045	$0.8280e-04$	2.016	$0.8310e-04$	2.017
320×320	$0.3125e-02$	$0.3168e+01$	$0.2230e-04$	2.016	$0.2065e-04$	2.004	$0.2070e-04$	2.005

only mesh ratio $\Delta t/H^2=\Delta t/h^{8/5}\leqslant C_1$ is required for some constant $C_1>0$. This restraint of mesh ratio is $h^{-2/5}$ multiple of that of explicit Galerkin scheme: $\Delta t/h^2\leqslant C^*$ for some constant $C^*>0$. To compare the results of each case, we take mesh ration $\Delta t/H^{5/2}=\Delta t/h^2=2.5$, where $2.5<\min\{(\frac{1}{20})^{-2/5},(\frac{1}{40})^{-2/5},(\frac{1}{80})^{-2/5},(\frac{1}{160})^{-2/5},(\frac{1}{320})^{-2/5}\}=\min\{3.31,4.37,5.77,7.61,10.05\}$.

In Table I, we give the L^2-norm error of $e_h=u-U$ for each case and fully implicit Galerkin method at time $t=0.5$.

And we present the L^∞-norm error of grids on the interface for each case at $t=0.5$. See Table II. Hereafter, *Implicit* means fully implicit Galerkin method and *L*(*R*)-*side* stands for the left(right)-hand side of Γ.

Remark 3
From Tables I and II, we know that the IM-PGDD scheme approximates the exact solution better than the fully implicit Galerkin method, having second-order convergence in h.

Next, we compare the CPU time cost of fully implicit Galerkin method and IM-PGDD for the time interval $[0,0.5]$. See Table III.

Remark 4
From Table III, we can see that the CPU time cost of IM-PGDD is smaller than that of implicit Galerkin scheme for each case and when the mesh is finer, IM-PGDD exhibits its superiority.

Example 2 (*EIM-PGDD scheme case*)**.**
We approximate (67) on $20\times 20, 40\times 40, 80\times 80, 160\times 160$ and 320×320 grids by the EIM-PGDD scheme. In these runs, we take $H^{9/2}=h^2$ to balance error accuracy. Since only mesh ratio $\Delta t/H^2=\Delta t/h^{8/9}\leqslant\tilde{C}_1$ is required for some constant $\tilde{C}_1>0$, which is $h^{-10/9}$ multiple of

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Table III. The CPU time cost for the time interval $[0, 0.5]$.

Grids	h	Implicit (s)	IM-PGDD (s)
20×20	0.5000e−01	2.04	0.51
40×40	0.2500e−01	15.59	4.42
80×80	0.1250e−01	287.81	70.74
160×160	0.6250e−02	552.39	142.36
320×320	0.3125e−02	831.74	209.27

Table IV. L^2-norm error for each case at $t=0.5$.

			Implicit		EIM-PGDD	
Grids	h	H/h	$\|e_h\|_{L^2}$	Rate	$\|e_h\|_{L^2}$	Rate
20×20	0.5000e−01	0.5282e+01	0.8065e−02		0.7884e−02	
40×40	0.2500e−01	0.7763e+01	0.2020e−02	1.997	0.1972e−02	1.999
80×80	0.1250e−01	0.1141e+02	0.5050e−03	2.000	0.4930e−03	2.000
160×160	0.6250e−02	0.1677e+02	0.1264e−03	1.998	0.1233e−03	2.000
320×320	0.3125e−02	0.2465e+02	0.3154e−04	2.003	0.3083e−04	2.000

Table V. L^∞-norm error of grids on Γ at $t=0.5$.

			Implicit		IM-PGDD			
Grids	h	H/h		Rate	L-side	Rate	R-side	Rate
20×20	0.5000e−01	0.5282e+01	0.6323e−02		0.4974e−02		0.4987e−02	
40×40	0.2500e−01	0.7763e+01	0.1574e−02	2.006	0.1247e−02	1.996	0.1259e−02	1.986
80×80	0.1250e−01	0.1141e+02	0.3721e−03	2.081	0.3034e−03	2.039	0.3054e−03	2.033
160×160	0.6250e−02	0.1677e+02	0.9018e−04	2.045	0.7515e−04	2.013	0.7558e−04	2.015
320×320	0.3125e−02	0.2465e+02	0.2230e−04	2.016	0.1875e−04	2.003	0.1884e−04	2.004

that of explicit Galerkin scheme: $\Delta t/h^2 \leqslant C^*$ for some constant $C^*>0$. We take mesh ration $\Delta t/H^{9/2}=\Delta t/h^2=20$, where $20<\min\{(\frac{1}{20})^{-10/9}, (\frac{1}{40})^{-10/9}, (\frac{1}{80})^{-10/9}, (\frac{1}{160})^{-10/9}, (\frac{1}{320})^{-10/9}\}=\min\{27.90, 60.27, 130.18, 281.20, 607.44\}$.

Table IV shows the L^2-norm error of $e_h=u-U$ for each case and the fully implicit Galerkin method at time $t=0.5$.

Table V lists the L^∞-norm error of grids on the interface for each case at $t=0.5$.

Table VI shows the comparison of the CPU time cost for the time interval $[0, 0.5]$.

Remark 5

From Tables IV and V, we can see that the EIM-PGDD scheme also approximates the exact solution better than fully implicit Galerkin method, having second-order convergence in h. From Table VI, we also can see that the CPU time cost of EIM-PGDD is smaller than that of implicit Galerkin scheme and a little more than that of IM-PGDD scheme.

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Table VI. The CPU time cost for the time interval $[0, 0.5]$.

Grids	h	Implicit (s)	EIM-PGDD (s)
20×20	0.5000e−01	2.04	1.25
40×40	0.2500e−01	15.59	5.24
80×80	0.1250e−01	287.81	87.54
160×160	0.6250e−02	552.39	184.65
320×320	0.3125e−02	831.74	239.72

Example 3 (*A parabolic equation in general form (1): IM-PGDD scheme case*).
We consider the following problem on $\Omega=[0,1]\times[0,1]$:

$$\begin{aligned} u_t-\nabla\cdot(A\nabla u)+cu &= f \quad \text{in } \Omega\times(0,T] \\ (A\nabla u)\cdot v &= 0 \quad \text{on } \partial\Omega\times(0,T] \\ u^0(x,y) &= (x-x^2)^2(y-y^2)^2 \quad \text{in } \Omega,\ t=0 \end{aligned} \tag{68}$$

where the coefficients are

$$A(x,y)=\begin{pmatrix} 2x^2+1 & xy \\ xy & 2y^2+1 \end{pmatrix}, \quad c(x,y,t)=x^2+y^2+1$$

and $f(x,y,t)$ is chosen so that the solution $u(x,y,t)=\mathrm{e}^{-t}(x-x^2)^2(y-y^2)^2$. See Figure 5.

We approximate (68) by the IM-PGDD scheme as same as Example 1. That is to say, using 4-node quadrilateral mesh on $20\times 20, 40\times 40, 80\times 80, 160\times 160$ and 320×320 grids, respectively; taking mesh ratio $\Delta t/H^{5/2}=\Delta t/h^2=2.5$.

The L^2-norm errors of $e_h=u-U$ for each case at time $t=0.5$ are given in Table VII.

Table VIII shows the L^∞-norm error of grids on the interface for each case at $t=0.5$.

Table IX shows the comparison of the CPU time cost for the time interval $[0, 0.5]$.

Remark 6
From Tables VII–IX, we can still get the same conclusions as that of Tables I–III, proving that the IM-PGDD scheme can be used for parabolic equation in general form (1).

Example 4 (*A parabolic equation in general form (1): EIM-PGDD scheme case*).
We approximate (68) by the EIM-PGDD scheme as same as Example 2. We use 4-node quadrilateral mesh on $20\times 20, 40\times 40, 80\times 80, 160\times 160$ and 320×320 grids, respectively and take mesh ratio $\Delta t/H^{9/2}=\Delta t/h^2=20$.

Table X shows the L^2-norm error of $e_h=u-U$ for each case and the fully implicit Galerkin method at time $t=0.5$.

Table XI lists the L^∞-norm error of grids on the interface for each case at $t=0.5$.

Table XII shows the comparison of the CPU time cost for the time interval $[0, 0.5]$.

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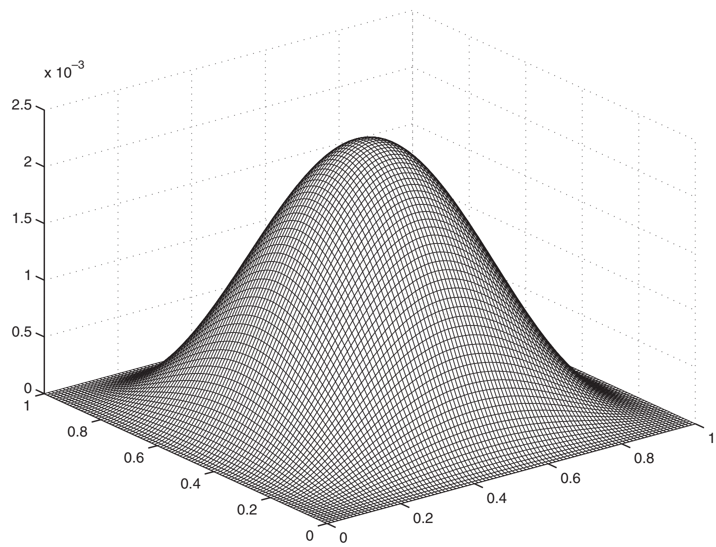

Figure 5. The solution $u(x,y,t)=e^{-t}(x-x^2)^2(y-y^2)^2$ at $t=0.5$.

Table VII. L^2-norm error for each case at $t=0.5$.

			Implicit		IM-PGDD	
Grids	h	H/h	$\|e_h\|_{L^2}$	Rate	$\|e_h\|_{L^2}$	Rate
20×20	0.5000e−01	0.1821e+01	0.9542e−02		0.8321e−02	
40×40	0.2500e−01	0.2091e+01	0.2407e−02	1.987	0.2086e−02	1.996
80×80	0.1250e−01	0.2402e+01	0.6050e−03	1.992	0.5223e−03	1.998
160×160	0.6250e−02	0.2759e+01	0.1516e−03	1.997	0.1306e−03	2.000
320×320	0.3125e−02	0.3168e+01	0.3791e−04	2.000	0.3264e−04	2.000

Table VIII. L^∞-norm error of grids on Γ at $t=0.5$.

			Implicit		IM-PGDD			
Grids	h	H/h		Rate	L-side	Rate	R-side	Rate
20×20	0.5000e−01	0.1821e+01	0.7256e−02		0.5849e−02		0.5898e−02	
40×40	0.2500e−01	0.2091e+01	0.1508e−02	2.267	0.1474e−02	1.989	0.1480e−02	1.995
80×80	0.1250e−01	0.2402e+01	0.3380e−03	2.158	0.3502e−03	2.074	0.3548e−03	2.060
160×160	0.6250e−02	0.2759e+01	0.8018e−04	2.076	0.8532e−04	2.037	0.8698e−04	2.028
320×320	0.3125e−02	0.3168e+01	0.1958e−04	2.034	0.2116e−04	2.012	0.2168e−04	2.004

Remark 7

From Tables X–XII, we can still get the same conclusions as that of Remark 5, proving that the EIM-PGDD scheme can be used for parabolic equation in general form (1).

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Table IX. The CPU time cost for the time interval $[0, 0.5]$.

Grids	h	Implicit (s)	IM-PGDD (s)
20×20	0.5000e−01	4.24	1.21
40×40	0.2500e−01	17.82	4.53
80×80	0.1250e−01	325.37	80.84
160×160	0.6250e−02	676.39	168.74
320×320	0.3125e−02	965.35	241.48

Table X. L^2-norm error for each case at $t=0.5$.

			Implicit		EIM-PGDD	
Grids	h	H/h	$\\|e_h\\|_{L^2}$	Rate	$\\|e_h\\|_{L^2}$	Rate
20×20	0.5000e−01	0.5282e+01	0.9542e−02		0.8027e−02	
40×40	0.2500e−01	0.7763e+01	0.2407e−02	1.987	0.2010e−02	1.998
80×80	0.1250e−01	0.1141e+02	0.6050e−03	1.992	0.5025e−03	2.000
160×160	0.6250e−02	0.1677e+02	0.1516e−03	1.997	0.1256e−03	2.000
320×320	0.3125e−02	0.2465e+02	0.3791e−04	2.000	0.3141e−04	2.000

Table XI. L^∞-norm error of grids on Γ at $t=0.5$.

			Implicit		IM-PGDD			
Grids	h	H/h		Rate	L-side	Rate	R-side	Rate
20×20	0.5000e−01	0.5282e+01	0.7256e−02		0.5632e−02		0.5686e−02	
40×40	0.2500e−01	0.7763e+01	0.1508e−02	2.267	0.1416e−02	1.992	0.1427e−02	1.994
80×80	0.1250e−01	0.1141e+02	0.3380e−03	2.158	0.3412e−03	2.053	0.3467e−03	2.041
160×160	0.6250e−02	0.1677e+02	0.8018e−04	2.076	0.8345e−04	2.032	0.8520e−04	2.025
320×320	0.3125e−02	0.2465e+02	0.1958e−04	2.034	0.2070e−04	2.011	0.2118e−04	2.008

Table XII. The CPU time cost for the time interval $[0, 0.5]$.

Grids	h	Implicit (s)	EIM-PGDD (s)
20×20	0.5000e−01	4.24	1.85
40×40	0.2500e−01	17.82	6.04
80×80	0.1250e−01	325.37	95.43
160×160	0.6250e−02	676.39	190.27
320×320	0.3125e−02	965.35	267.42

6. CONCLUSIONS AND PERSPECTIVES

We have presented Galerkin domain decomposition procedures for parabolic equations on rectangular domain. IM-PGDD and EIM-PGDD schemes have been constructed by the integral mean method to approximate explicitly the flux on the inter-domain boundary. They are also

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explicit/implicit schemes. Thus, the parallelism can be achieved by the use of these procedures. These procedures are conservative both in the sub-domains and across inter-boundaries. Mesh ratio $\Delta t/H^2 \leqslant C$ for some constant $C>0$ is still needed for these procedures to preserve stability, but less severe than that for fully explicit methods. To bound L^2-norm error estimates, new elliptic projections are established and analyzed. Compared with [9], these L^2-norm error estimates avoid the loss of $H^{-1/2}$ factor. Moreover, L^2-norm error estimate of the EIM-PGDD scheme has higher order of H than that of the IM-PGDD scheme. Numerical experiments have confirmed our theoretical results.

The main aim of this paper is to present explicit flux calculation on the inter-domain boundary by integral mean method. We can extend our method combined with dynamic grid modification to parabolic equations. The results for this case will be presented in a forthcoming paper.

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